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# AN ALTERNATE DERIVATION OF
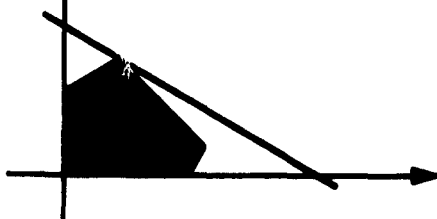# THE POLLACZEK-KHINTCHINE FORMULA

by

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# OPERATIONS RESEARCH CENTER

### INSTITUTE OF ENGINEERING RESEARCH

# UNIVERSITY OF CALIFORNIA - BERKELEY

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THE POLLACZEK-KHINTCHINE FORMULA


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## THE POLLACZEK-KHINTCHINE FORMULA

A derivation of the Pollaczek-Khintchine formula follows from the results obtained by John D. C. Little [1] in his proof that

(1)
$$L = \lambda W$$

where $\lambda$ is the average rate of arrivals into a single-channel queue W is the average delay, and L is the average number in the queueing system at a random instant of time. It is assumed that the queueing process is strictly stationary.

In this derivation we also assume that inter-arrival times and service times are independently sampled positive random variables. We use the notation

$\pi_0$ = Pr (empty queueing system at a random instant of time)

$\lambda$ = Average arrival rate (a number)

$\frac{1}{\mu}$ = Average service time (a number)

$\sigma^2$ = Variance of the service time (a number)

X = Delay ($\geq$ 0) in queue of a customer arriving at a random instant of time (a random variable)

$W_x$ = Expected value of X (a number)

Y = Time ($\geq$ 0) required to complete the service of a customer in service at a random instant of time (a random variable)

$W_y$ = Expected value of Y (a number)

Z = Time ($\geq$ 0) to service the customers in queue at a random instant of time (a random variable)

$W_z$ = Expected value of Z

From the definitions it follows that

(2a)                                    $X = Y + Z$

and that

(2b)                                    $W_x = W_y + W_z$

Introducing the subscript x to denote averages in queue (i.e., exclusive
of services) in Equation (1) and the assumption that service times are
independent of the number in queue, the average time to service the number
in queue at a random instant of time is

(3)                                    $W_z = \frac{1}{\mu} (\lambda W_x)$

If the service facility is empty at the instant a customer arrives, $Y = 0$.
If the service facility is busy, the expected value of the remaining service
time of the customer in service is the expected value of the length-biased
sampling distribution of service times.[*] Unconditionally,

(4)                  $W_y = (1 - \pi_0) \frac{\mu(\sigma^2 + \mu^{-2})}{2} = \frac{\lambda(\sigma^2 + \mu^{-2})}{2}$

Substituting (3) and (4) into (2b) and solving for $W_x$ gives the Pollaczek-
Khintchine formula,

(5)                                    $W_x = \frac{\lambda(\sigma^2 + \mu^{-2})}{2(1 - \lambda/\mu)}$

This derivation can be extended to those cases where mixed streams feed
the service channel with different priorities of service.

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[*]Reference 2, page 65.

# REFERENCES

1. Little, John D.C., "A Proof for the Queueing Formula: $L = \lambda W$," Operations Research, Vol. 9, (1961), pp. 383.

2. Cox, D.R., "Renewal Theory," Methuen Monograph, John Wiley and Sons, Inc., 1962.

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